Dynamics, symmetry and geometry

Part I

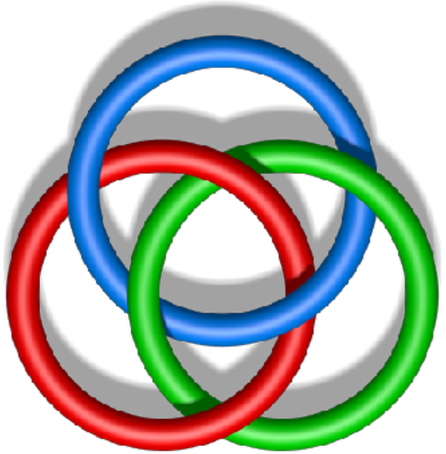
Qi Zhou

Department of Physics and Astronomy, Purdue University

Croucher Summer Course “Ultracold Atom Physics 2024”, HK

- A simple question in quantum mechanics about distance
- A concept in quantum computation : circuit complexity
- Geometric descriptions of quantum dynamics

Symmetry



Dynamics **Geometry**

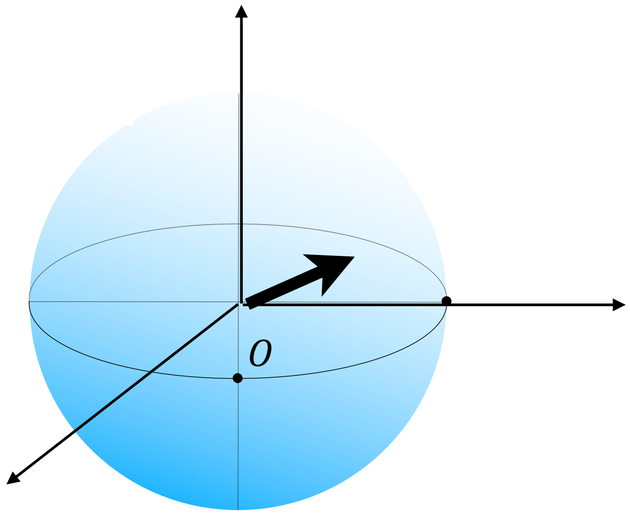
Quantum evolution: a central task in quantum mechanics

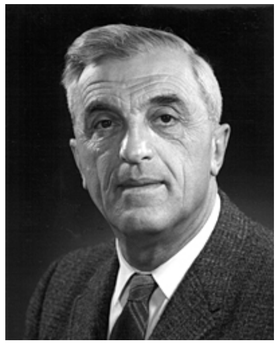
Schrödinger equation $i\partial_t \Psi = H(t)\Psi(t)$ Mathematical description

How to visualize quantum dynamics?

The most fundamental example:

Bloch sphere

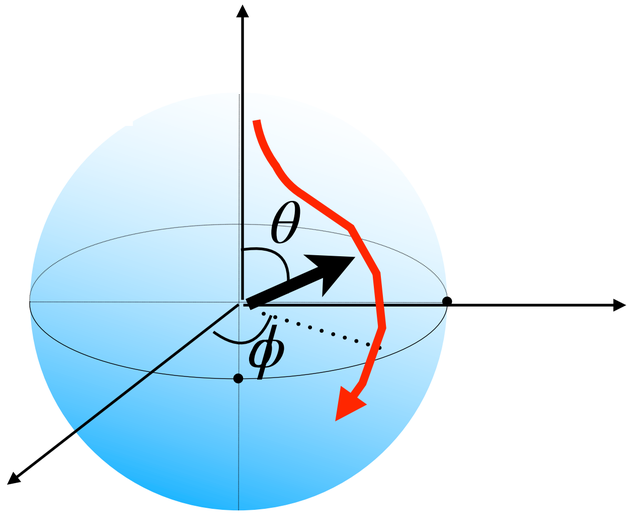




Felix Bloch

Textbook results

$$H(t) = \vec{B} \cdot \vec{S}$$



$$\Psi = \begin{pmatrix} e^{-i\frac{\phi}{2}} \cos\frac{\theta}{2} \\ e^{i\frac{\phi}{2}} \sin\frac{\theta}{2} \end{pmatrix}$$

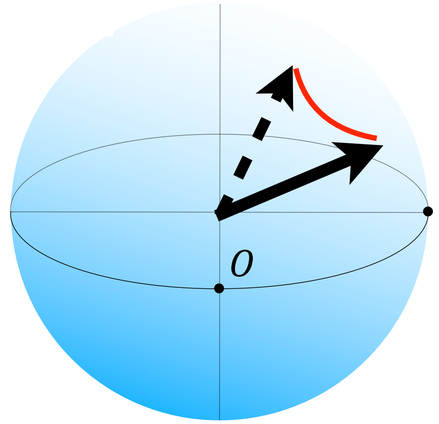
$i\partial_t \Psi = H(t)\Psi(t)$ Trajectory on a Bloch sphere $\left(\theta(t), \phi(t)\right)$

Bloch **sphere** Why a sphere?

$$\Psi(\theta,\phi)=\begin{pmatrix} e^{-i\frac{\phi}{2}}\cos\frac{\theta}{2} \\ e^{i\frac{\phi}{2}}\sin\frac{\theta}{2}\end{pmatrix}$$

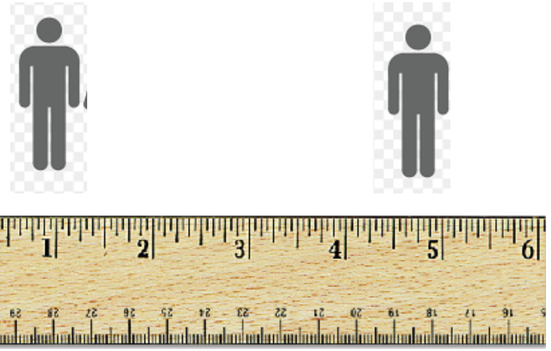
$$\langle S_z\rangle=\cos\theta \qquad \langle S_x\rangle=\sin\theta\cos\phi \qquad \langle S_y\rangle=\sin\theta\sin\phi$$

$$\langle S_z\rangle^2+\langle S_y\rangle^2+\langle S_z\rangle^2=1$$

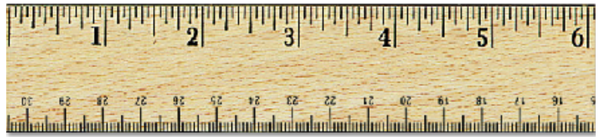


But… $ds^2=d\theta^2+\sin^2\theta d\phi^2$

How about the distance between two wavefunctions?



$|\Psi\rangle$ $|\Psi'\rangle$



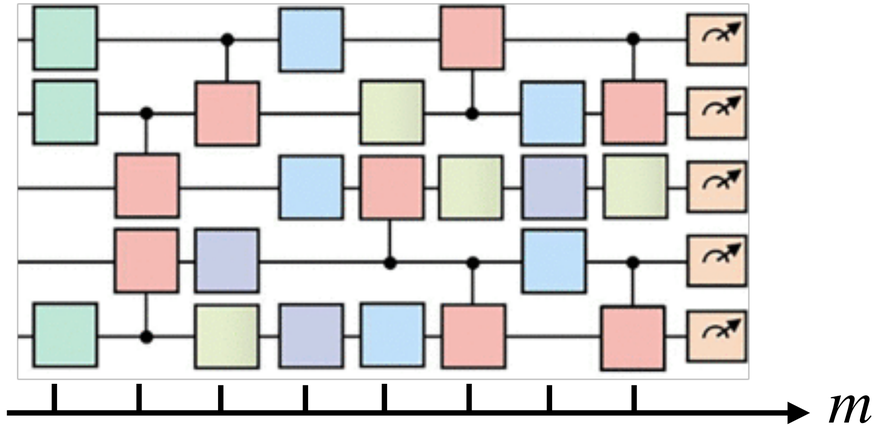
Fidelity $ds_F^2=1-|\langle\Psi'|\Psi\rangle|^2$ $ds_F^2=d\theta^2+\sin^2\theta d\phi^2$

$\Psi'=\Psi(\theta+d\theta,\phi+d\phi)$ Any other more physical means?

A tool in quantum computation to help

Circuit complexity

M. A. Nielsen, et al., Quantum computation as geometry, Science 311, 1133 (2006)



$G_1, G_2, \ldots, G_N$ Generators are fixed

A gate: $e^{-i\sum_{i=1}^{N} Y_i(m) G_i}$

How to design the **optimal** quantum algorithm to reach the target

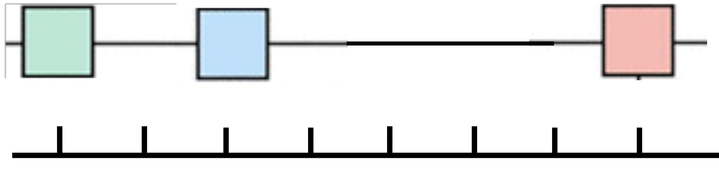
Optimal: the least **cost**

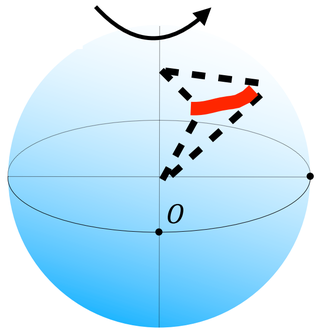
An intuitive picture: operating a single gate requires a certain cost

The total cost needs to be minimized

We need to quantitatively define the **cost**

Example: A single qubit, a single generator S_z





$Y(m)$ Rotation angle in the m-th step

$e^{-iY(m)S_z}$ A gate in the m-th step: a rotation about the z-axis

Lager $|Y(m)|$, larger cost

You need to wait longer
or your advisor needs to pay a larger electricity bill

Cost in each step: $f(m) = |Y(m)|$

Total cost $F = \sum_m f(m) = \sum_m |Y(m)|$



More generic cases $e^{-i\sum_{i=1}^{N} Y_i(m)G_i}$

Cost in each step: $f(m) = \sum_{i=1}^{N} |Y_i(m)|$ Cost function: $F = \sum_{m} f(m)$

Other cost functions are also allowed

$f(m) = \sqrt{\sum_{i=1}^{N} Y_i^2(m)}$ cost of applying generators are not independent

$f(m) = \sqrt{\sum_{i=1}^{N'<N} Y_i^2(m)}$ No cost to apply certain generators at all

A quantum algorithm is a discrete quantum evolution

Continuum limit: Discrete label m is replaced by a continuous variable τ

$$H(\tau) = \sum_{i=1}^{N} \xi_i(\tau) G_i$$

$\Psi(\tau) = U(\tau,0)\Psi(0)$ Propagator $U(\tau,0) = Te^{-i\int_0^\tau H(\tau')d\tau'}$

Cost function $F = \int_0^\tau d\tau' f(\tau')$

$$f(\tau') = \sum_{i}^{m} |\,\xi_i(\tau')\,|, \quad f(\tau') = \sqrt{\sum_i \xi_i^2(\tau')}, \ldots$$

Example: $H(\tau) = \sum_{i=1}^{3} \xi_i(\tau) G_i$ $(G_1, G_2, G_3) = (S_x, S_y, S_z)$

$\vec{\xi} = (\xi_1, \xi_2, \xi_3)$ A magnetic field

Cost function $f(\tau') = \sqrt{\sum_{i=1}^{3} \xi_i^2(\tau')}$ $F = \int_0^{\tau} d\tau' f(\tau')$

Physical meaning:

If the strength of the magnetic field is fixed $B = \sqrt{\sum_{i=1}^{3} \xi_i^2(\tau')} = \text{constant}$

$F = \text{constant} \times \tau$ Cost = time !



Least cost: least time to reach the target state!

Optimal algorithm: the fastest approach

Why a constant B ?

Fair comparison: larger B , faster rotation, less time

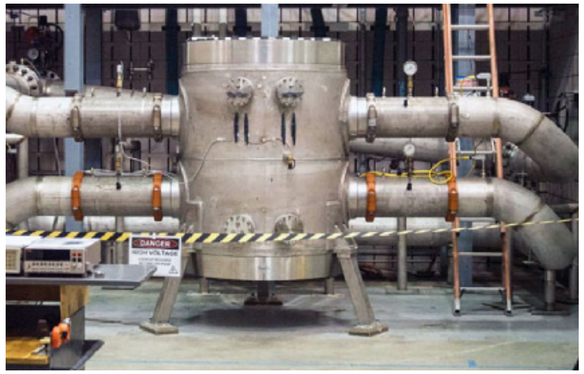
To compare different algorithms or approaches, we need to choose the same B

In any case, there is an upper limit of B one can access in a lab

MagLab Reclaims Record for Strongest Resistive Magnet

Published August 22, 2017

https://nationalmaglab.org/news-events/news/strongest-resistive-magnet



41.4-tesla

China has set a new world record for the strongest steady magnetic field

SEPTEMBER 7, 2022

https://scitechdaily.com/1400000-times-stronger-than-earths-new-record-for-strongest-steady-magnetic-field/



45.22 teslas

Defining a cost function allows us to unfold the geometry

A geometric interpretation of circuit complexity

M. A. Nielsen, et al., Quantum computation as geometry, Science 311, 1133 (2006)

Example:

$$H(\tau) = \sum_{i=1}^{3} \xi_i(\tau) G_i \qquad (G_1, G_2, G_3) = (S_x, S_y, S_z)$$

Generators of SU(2)

$$[S_i, S_j] = i\epsilon_{ijk} S_k$$

$$\Psi(\tau) = U(\tau,0)\Psi(0) \qquad U(\tau,0) = Te^{-i\int_0^\tau H(\tau')d\tau'}$$

Parametrized by $U(\phi,\theta,\varphi) = e^{-i\phi S_z} e^{-i\theta S_y} e^{-i\varphi S_z}$

$$U(\phi,\theta,\varphi)\begin{pmatrix}1\\0\end{pmatrix} = e^{-i\frac{\varphi}{2}}\begin{pmatrix} e^{-i\frac{\phi}{2}}\cos\frac{\theta}{2} \\ e^{i\frac{\phi}{2}}\sin\frac{\theta}{2}\end{pmatrix}$$

$$U(\phi,\theta,\varphi) = e^{-i\phi S_z} e^{-i\theta S_y} e^{-i\varphi S_z}$$

$$i\frac{dU}{d\tau}U^{-1} = \sum_{i=x,y,z} X_i\left(\phi,\theta,\varphi,\frac{d\phi}{d\tau},\frac{d\theta}{d\tau},\frac{d\varphi}{d\tau}\right)S_i$$

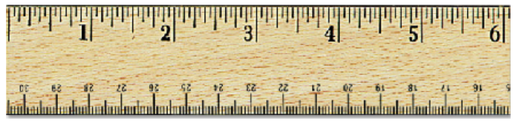
Excise: work out the explicit expressions

$$U(\tau,0) = Te^{-i\int_0^\tau H(\tau')d\tau'}$$

$$i\frac{dU}{d\tau}U^{-1} = H(\tau) = \xi_x S_x + \xi_y S_y + \xi_z S_z$$

Define a line element $ds = d\tau\sqrt{\xi_x^2 + \xi_y^2 + \xi_z^2}$

$$F = \int ds$$



Length = Time



(Constant magnetic field)

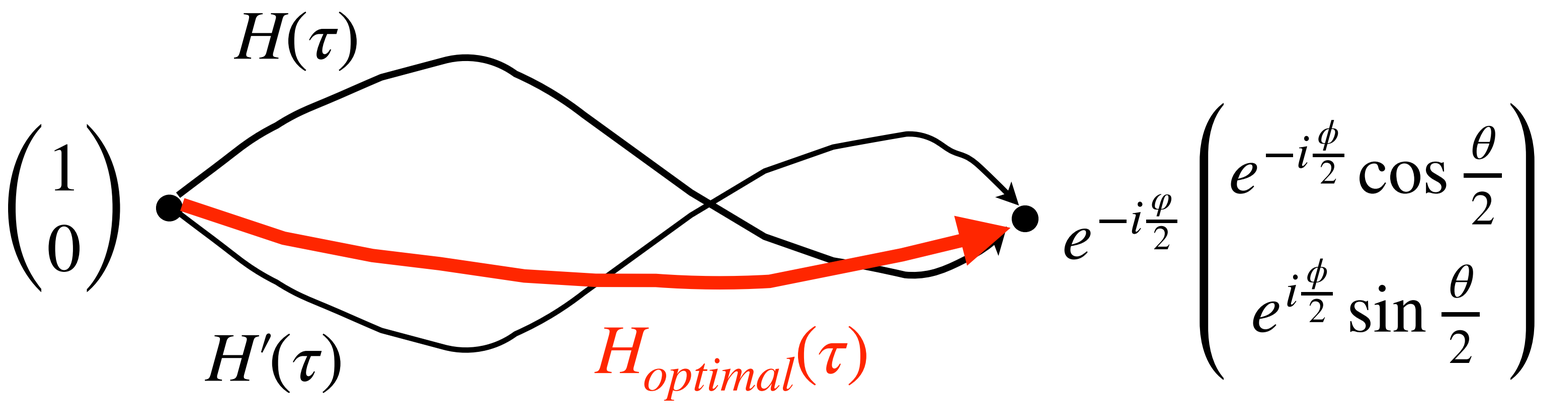
$$ds^2 = d\theta^2 + d\phi^2 + d\varphi^2 + 2\cos\theta d\varphi d\phi$$

Three-dimensional sphere (3-sphere)

Any quantum dynamics is mapped to a trajectory on a 3-sphere

$$ds^2 = d\theta^2 + d\phi^2 + d\varphi^2 + 2\cos\theta d\varphi d\phi$$

Length on a 3-sphere= time spent to reach the final state



$$\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$e^{-i\frac{\varphi}{2}} \begin{pmatrix} e^{-i\frac{\phi}{2}} \cos\frac{\theta}{2} \\ e^{i\frac{\phi}{2}} \sin\frac{\theta}{2} \end{pmatrix}$$

Geodesic on a 3-sphere= Fastest trajectory to reach the final state

From 3-sphere to 2-sphere

$$e^{-i\frac{\varphi}{2}}\begin{pmatrix} e^{-i\frac{\phi}{2}}\cos\frac{\theta}{2} \\ e^{i\frac{\phi}{2}}\sin\frac{\theta}{2} \end{pmatrix}$$

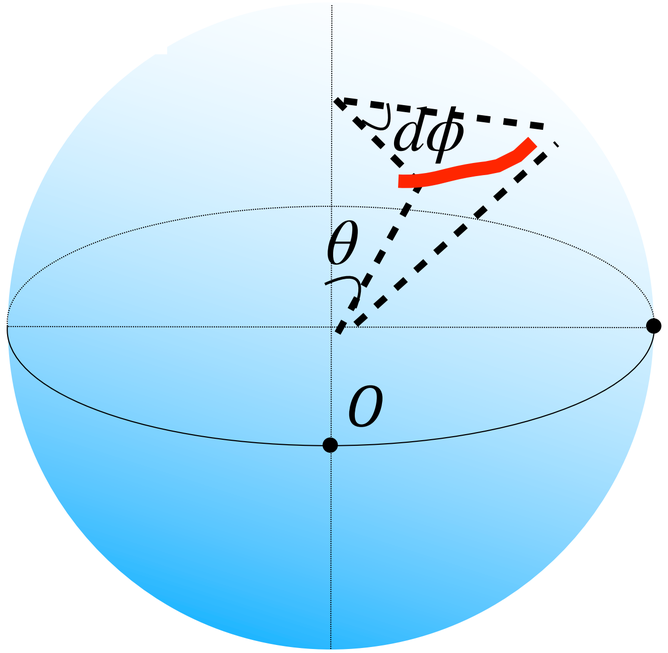
Different φ : the same state

$$ds^2 = d\theta^2 + d\phi^2 + d\varphi^2 + 2\cos\theta d\varphi d\phi$$

Consider ds as a function of $d\varphi$: $ds\left(d\varphi\right)$

Choose $d\varphi$ that gives rise to min ds

$$d\varphi = -\cos\theta d\phi$$



$$ds^2 = d\theta^2 + \sin^2\theta d\phi^2$$

Other cost functions

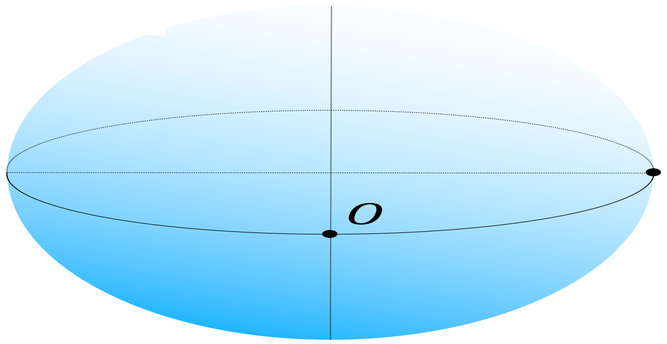
$$ds = d\tau\sqrt{\alpha_x\xi_x^2 + \alpha_y\xi_y^2 + \alpha_z\xi_z^2}$$

Fixing $\alpha_x\xi_x^2 + \alpha_y\xi_y^2 + \alpha_z\xi_z^2$ in comparisons

Example:

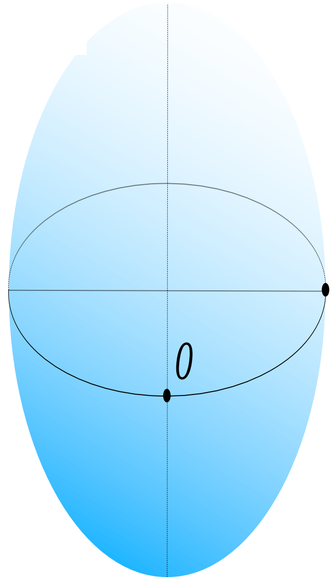
$$\alpha_x = \alpha_y < \alpha_z$$

More cost when rotating about z



$$\alpha_x = \alpha_y > \alpha_z$$

Less cost when rotating about z



Symmetry

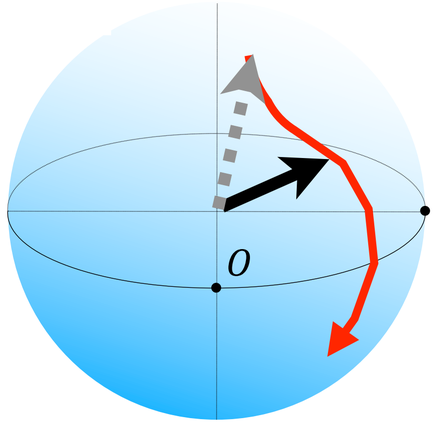
SU(2)

$$[S_x, S_y] = iS_z$$

$$[S_y, S_z] = iS_x$$

$$[S_z, S_x] = iS_y$$

Geometry



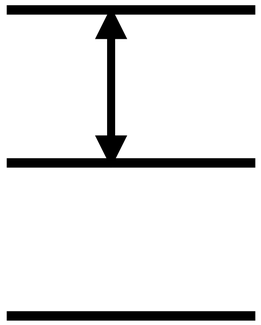
Dynamics

$$\Psi(\tau) = \begin{pmatrix} e^{-i\frac{\phi(\tau)}{2}} \cos\frac{\theta(\tau)}{2} \\ e^{i\frac{\phi(\tau)}{2}} \sin\frac{\theta(\tau)}{2} \end{pmatrix}$$

Another example:

$$H(\tau) = \sum_{i=0}^{2} \xi_i(\tau) K_i$$

$$\sqrt{\xi_0^2 - \xi_1^2 - \xi_2^2}$$



$$\sqrt{\xi_0^2 + \xi_1^2 + \xi_2^2}$$

$$H(\tau) = \sum_{i=0}^{2} \xi_i(\tau) S_i$$

$$[K_1, K_2] = -iK_0$$

$$[K_0, K_1] = iK_2$$

$$[K_2, K_0] = iK_1$$

Field strength

$$\tilde{B} = \sqrt{\xi_0^2 - \xi_1^2 - \xi_2^2}$$

Field strength

$$B = \sqrt{\xi_0^2 + \xi_1^2 + \xi_2^2}$$

$$U = e^{-iK_0(\varphi-\psi)} e^{-2iK_1\eta} e^{-iK_0(\varphi+\psi)}$$

Define line element

$$ds = \sqrt{-\xi_0^2 + \xi_1^2 + \xi_2^2}\, d\tau/(2\hbar)$$

$$ds^2 = \mathrm{d}\eta^2 - \cosh^2(\eta)\mathrm{d}\varphi^2 + \sinh^2(\eta)\mathrm{d}\psi^2$$

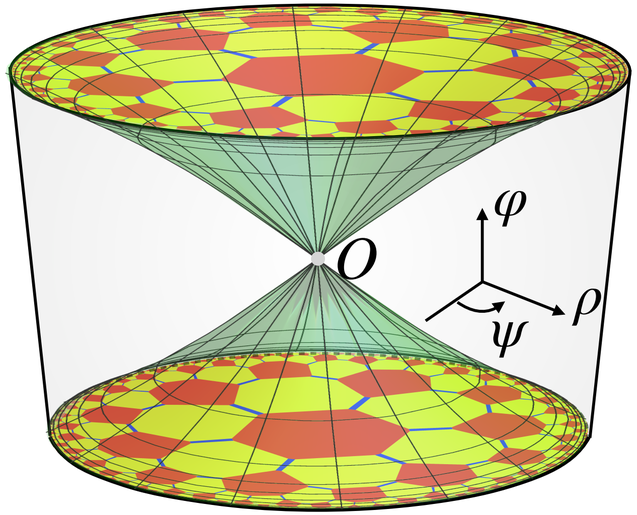
$$ds^2 = \mathrm{d}\eta^2 - \cosh^2(\eta)\mathrm{d}\varphi^2 + \sinh^2(\eta)\mathrm{d}\psi^2$$

$d\varphi = 0$

2D projection

$$ds^2 = \mathrm{d}\eta^2 + \sinh^2(\eta)\mathrm{d}\psi^2$$

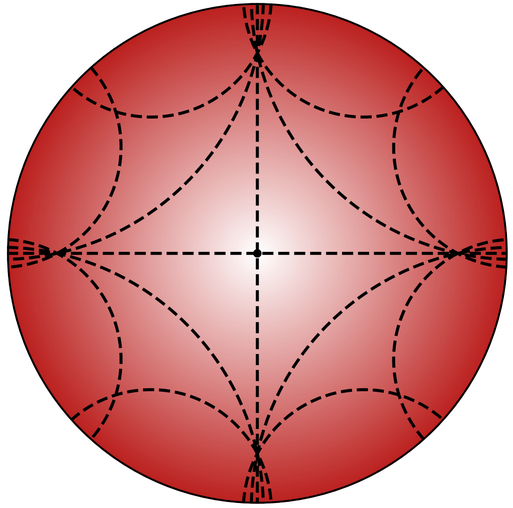
(2+1)-d Anti-de Sitter (AdS_{2+1})



Light cone

$$\eta = 2\mathrm{arctanh}(\rho)$$

Poincare disk



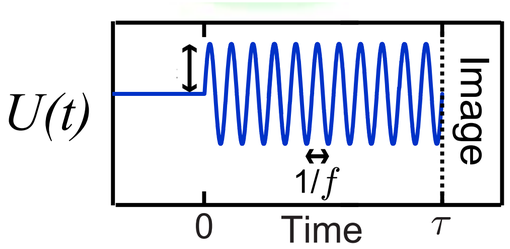
Examples of quantum systems with SU(1,1) dynamical symmetry

Bogoliubov theory for weakly interacting bosons

$$H = \sum_{\vec{k}} \epsilon_{k0} c_k^\dagger c_k + U_0 \sum_{\vec{k},\vec{k'},\vec{q}} c^\dagger_{\vec{q}+\vec{k}} c^\dagger_{\vec{q}-\vec{k}} c_{\vec{q}-\vec{k'}} c^\dagger_{\vec{q}+\vec{k'}}$$

condensation at $\vec{k_0} = 0$

$$H = \sum_{\vec{k}} \epsilon_k c_k^\dagger c_k + U \sum_{\vec{k}} (c^\dagger_{\vec{k}} c^\dagger_{-\vec{k}} + h.c.) \qquad U = U_0 N_0$$



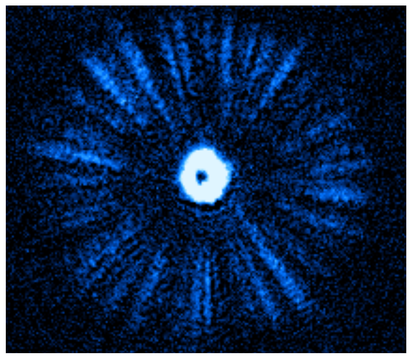
$$H = \sum_{\vec{k}} \epsilon_k c_k^\dagger c_k + U \cos(\omega t) \sum_{\vec{k}} (c^\dagger_{\vec{k}} c^\dagger_{-\vec{k}} + h.c.) = \sum_{\vec{k}} H_{\vec{k}}$$

$$K_0 = \frac{1}{2}(c^\dagger_{\vec{k}} c_{\vec{k}} + c_{-\vec{k}} c^\dagger_{-\vec{k}})$$

$$K_1 = \frac{1}{2}(c^\dagger_{\vec{k}} c^\dagger_{-\vec{k}} + c_{\vec{k}} c_{-\vec{k}})$$

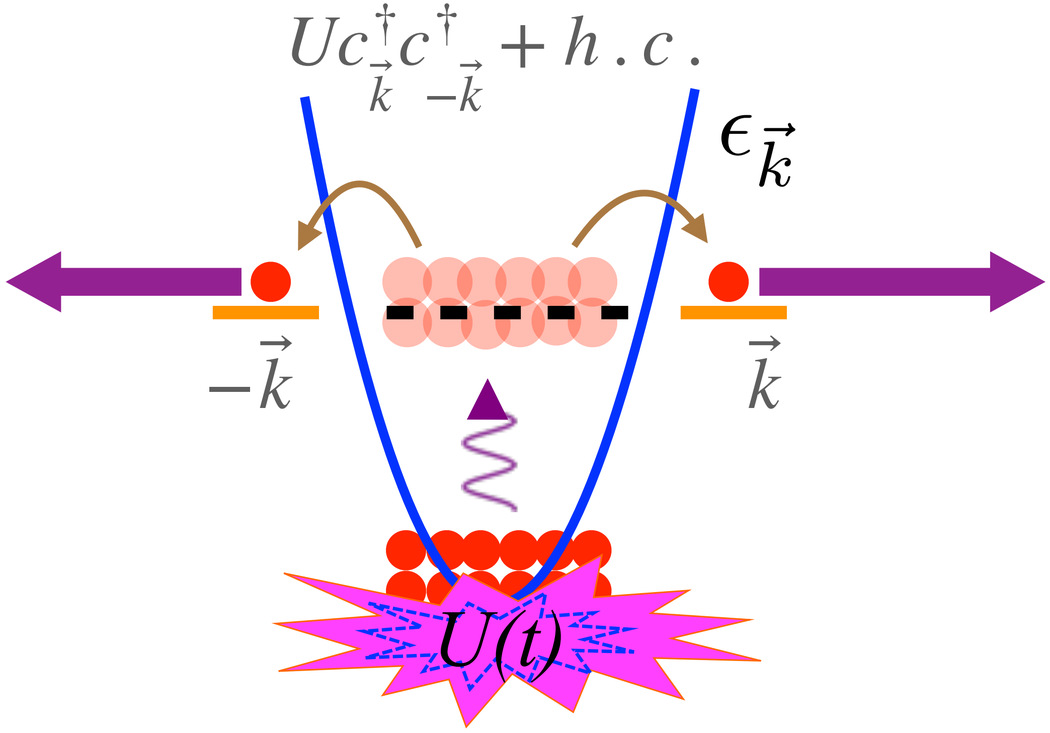
$$K_2 = \frac{1}{2i}(c^\dagger_{\vec{k}} c^\dagger_{-\vec{k}} - c_{\vec{k}} c_{-\vec{k}})$$

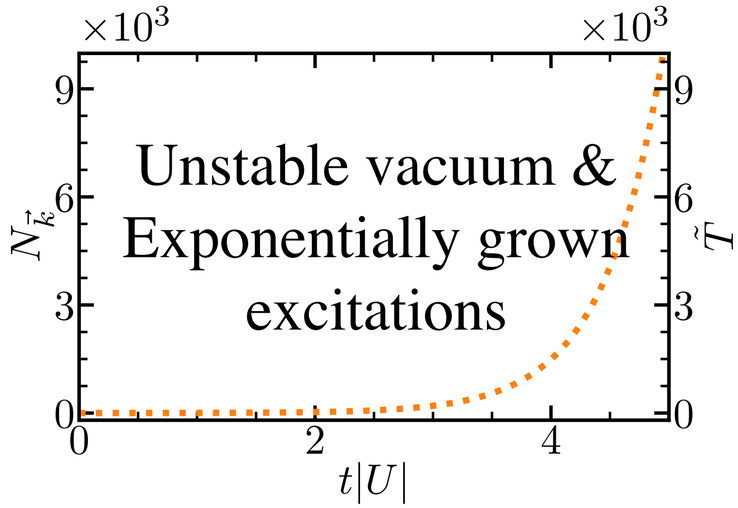
$$H_{\vec{k}} = \sum_{i=0}^{2} \xi_i K_i$$



Fireworks: correlated pairs

Clark, et al., Nature 551, 356 (2017)





Dynamical instability
Parametric amplification (AMO/CM)
Two-mode squeezing

$$|\Psi(t)\rangle = \sum_n \left(-i\tanh(\frac{Ut}{\hbar})\right)^n |n\rangle_{\vec{k}} |n\rangle_{-\vec{k}}$$

Thermofield double state (HEP)

$$\rho = \mathrm{Tr}_{-\vec{k}} |\Psi(t)\rangle\langle\Psi(t)|$$

Reduced density matrix:
thermalized

$$k_b T/E_{\vec{k}} = -\ln^{-1}[\tanh(2Ut)]/2$$

time ⟷ ***T***emperature

Dynamics, symmetry and geometry

Part II

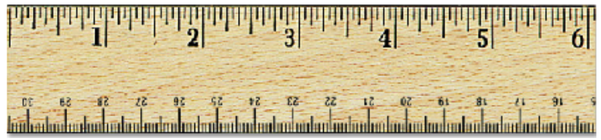
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Croucher Summer Course “Ultracold Atom Physics 2024”, HK

- A simple question in quantum mechanics about distance between two wavefunctions

$|\Psi\rangle$ $|\Psi'\rangle$



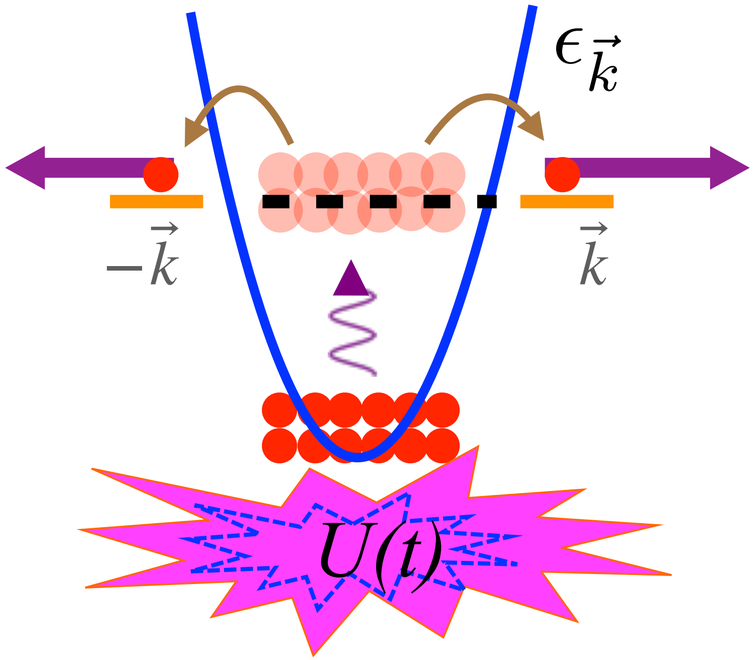
- A concept in quantum computation : circuit complexity

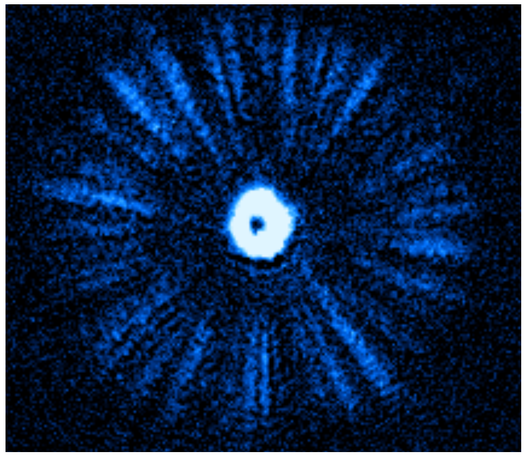
$$i\frac{dU}{d\tau}U^{-1} = H(\tau)$$



- **Geometric descriptions of quantum dynamics**

Boson Fireworks

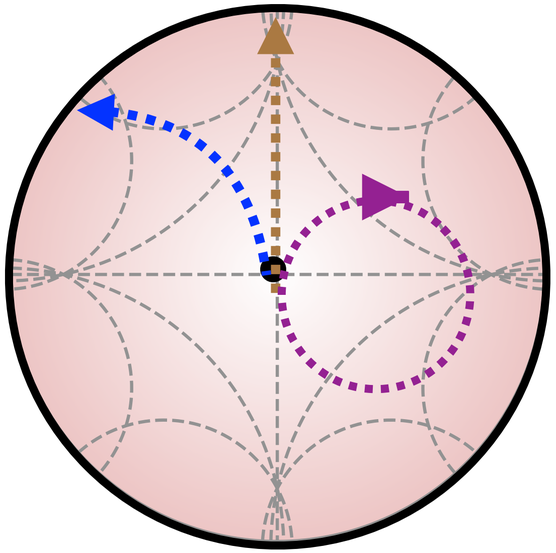




Clark, et al., Nature 551, 356 (2017)

Dynamical instability/
Parametric amplification

Poincaré disk: Visualization of more generic dynamics

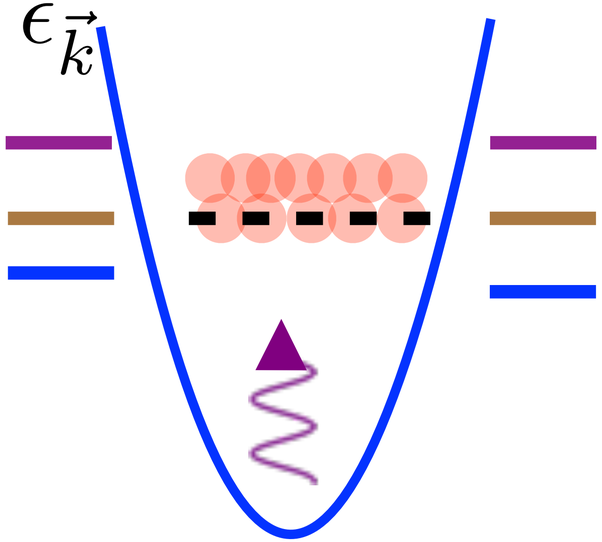


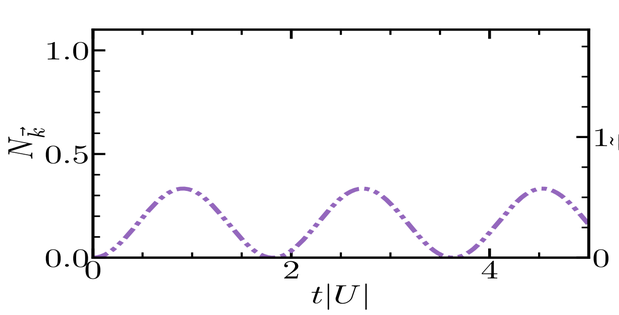
C Lyu, C Lv, Q Zhou PRL125, 253401 (2020)

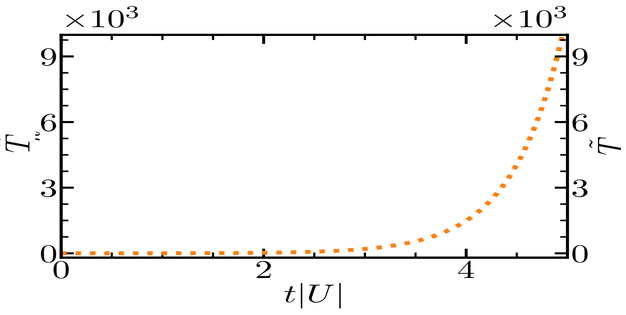
points on Poincare disk: SU(1,1) coherent states

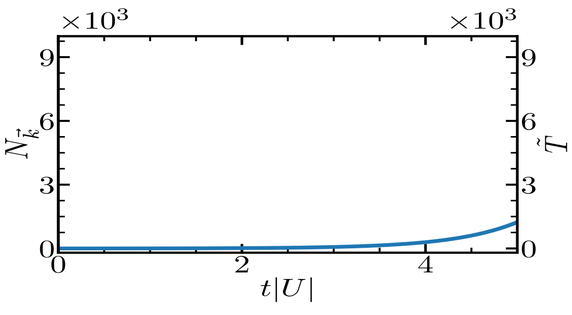
$$|z\rangle = \sqrt{1-|z|^2}\sum_n z^n |n\rangle_{\vec{k}} |n\rangle_{-\vec{k}} \qquad z = x + iy$$

$$|\Psi(t)\rangle = Te^{-i\int H(\xi_0,\xi_1,\xi_2)dt'}|\Psi(0)\rangle$$ trajectories on Poincaré disk







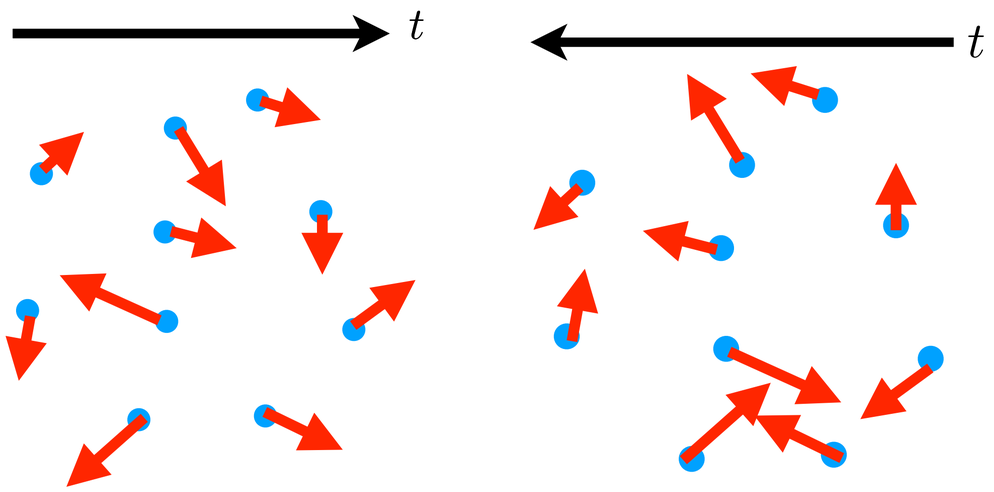


Stable modes: closed loops

Resonance modes: geodesics

Unstable modes: open trajectories

Geometrization: more than visualization



$H \to -H$

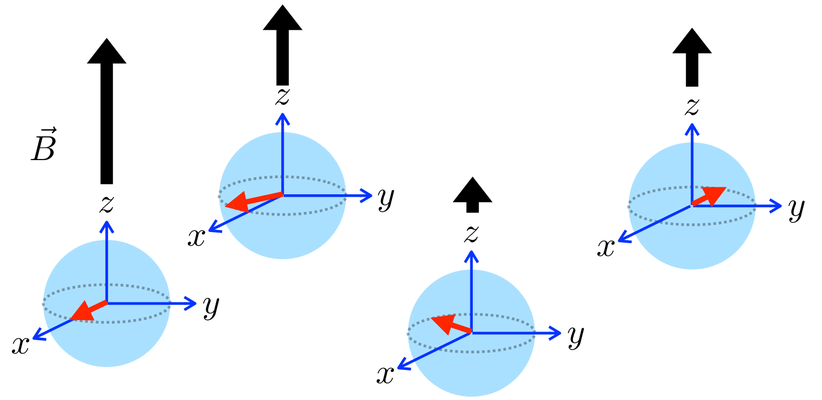
$K \to -K$

$U \to -U$

$U \to -V$

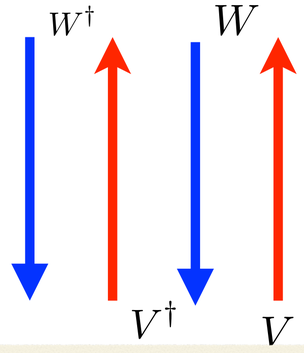
Difficult

Overcoming dephasing



Out-of-time-order correlator

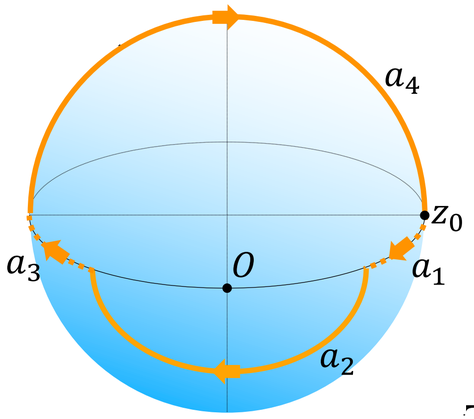
$$e^{iHt}W^{\dagger}(0)e^{-iHt}V^{\dagger}(0)e^{iHt}W(0)e^{-iHt}V(0)$$

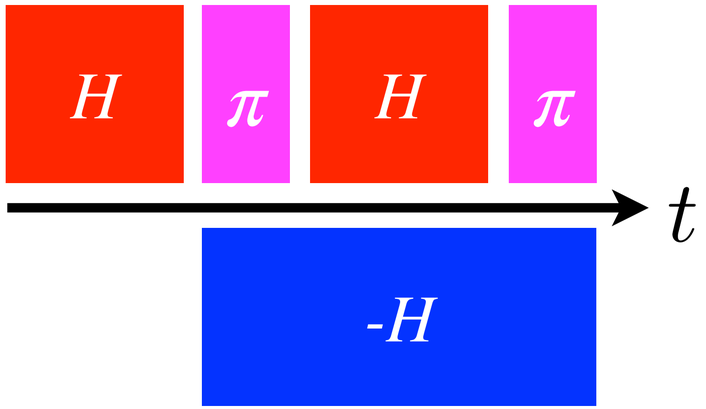


Backward evolution

forward evolution

Spin echo

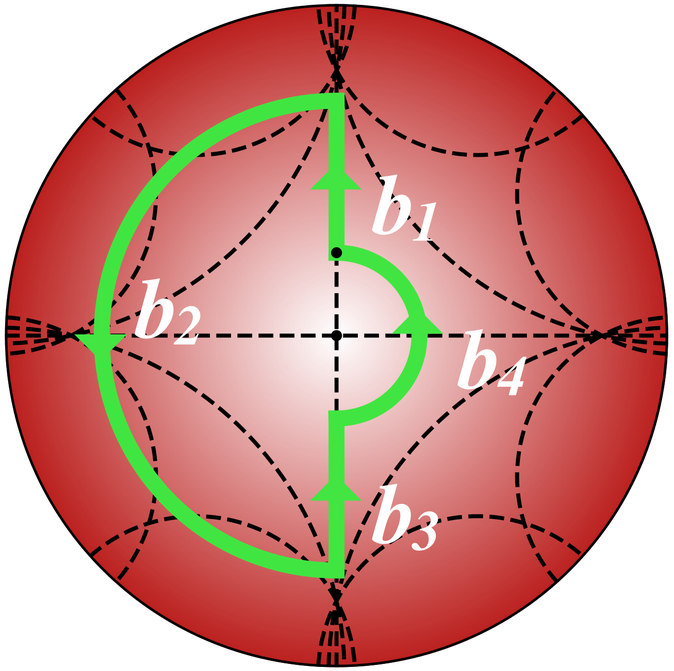




Two π-pulse + forward evolution=backward evolution

How about other systems not obeying su(2) algebra?

SU(1,1) echoes Reversing the dynamical instability

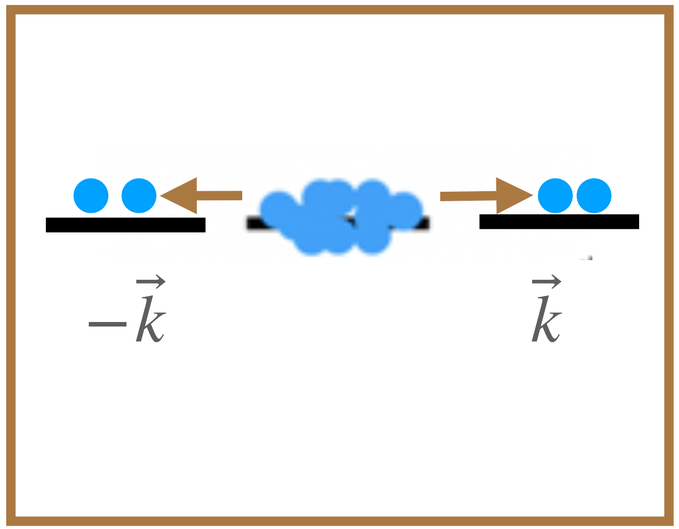


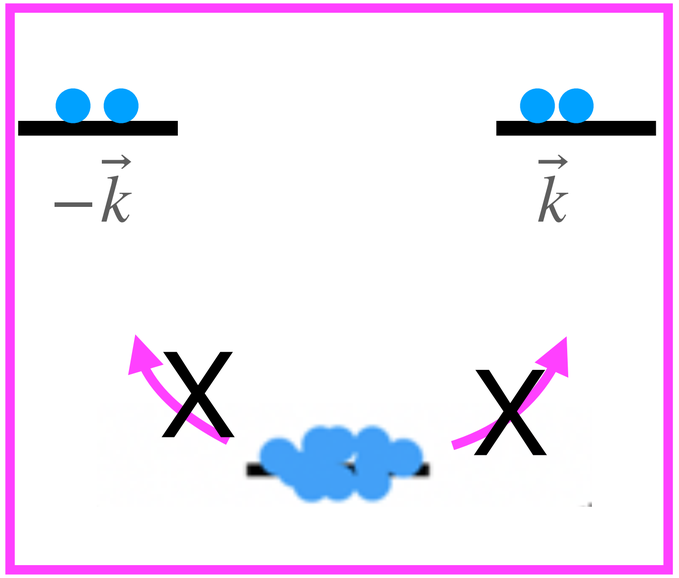
$$b_1: \quad H_1 = \frac{1}{2}(c^\dagger_{\vec{k}} c^\dagger_{-\vec{k}} + c_{\vec{k}} c_{-\vec{k}})$$

$$b_2: \quad H_2 = \frac{1}{2}(c^\dagger_{\vec{k}} c_{\vec{k}} + c_{-\vec{k}} c^\dagger_{-\vec{k}})$$

$$b_3: \quad H_1 = \frac{1}{2}(c^\dagger_{\vec{k}} c^\dagger_{-\vec{k}} + c_{\vec{k}} c_{-\vec{k}})$$

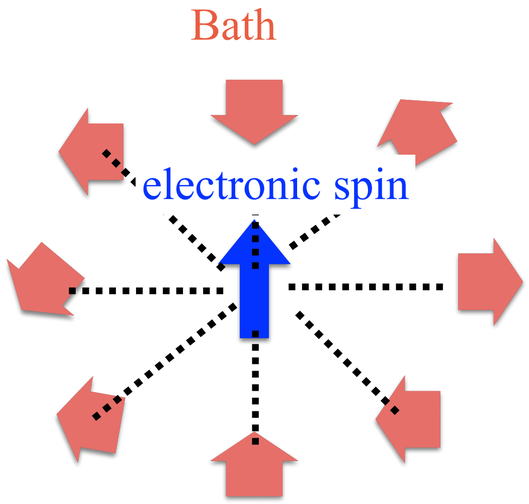
$$b_4: \quad H_2 = \frac{1}{2}(c^\dagger_{\vec{k}} c_{\vec{k}} + c_{-\vec{k}} c^\dagger_{-\vec{k}})$$





More than reversing quantum dynamics

Spin echo



Powerful tool to detect **many-body physics of the bath**

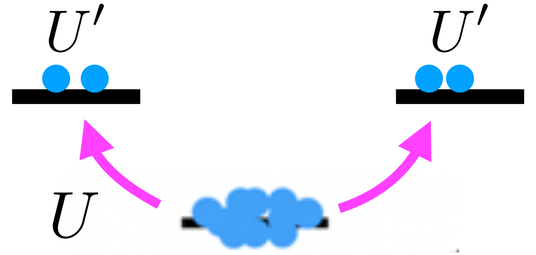
Quantum criticality, Lee-Yang zeros…

PRL 96, 140604 (2006); PRL 109, 185701(2012); PRL114, 010601(2015)…

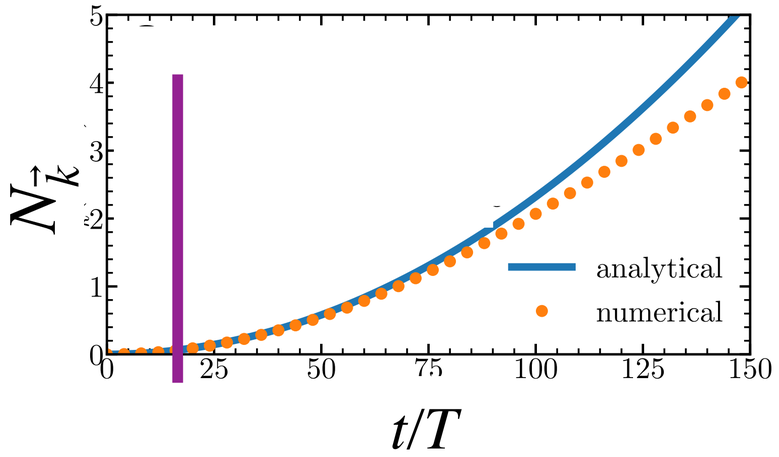
SU(1,1) echoes

perturbations breaking the symmetry → echoes cannot recover the initial state

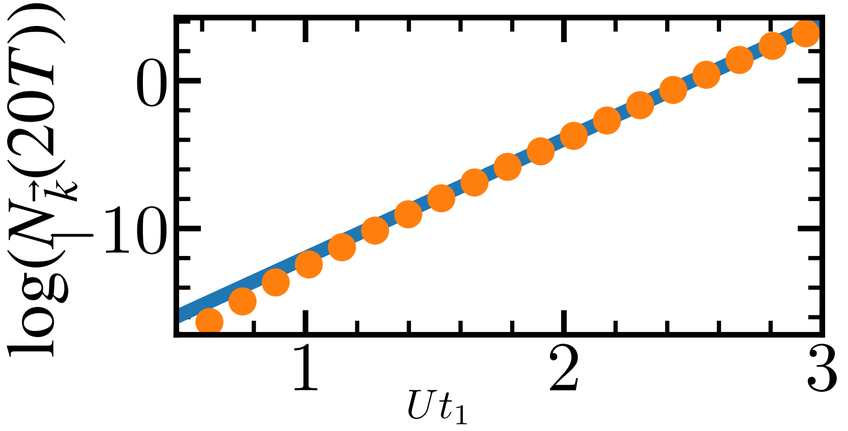
Quantum enhanced sensor



Interactions in excited states

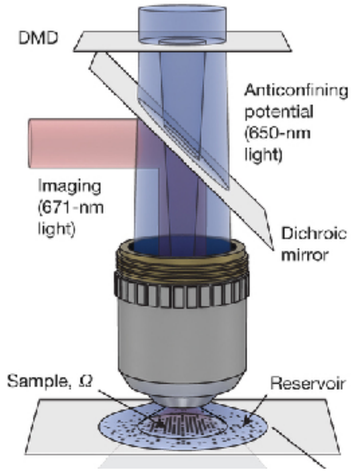


Exponentially depend on U



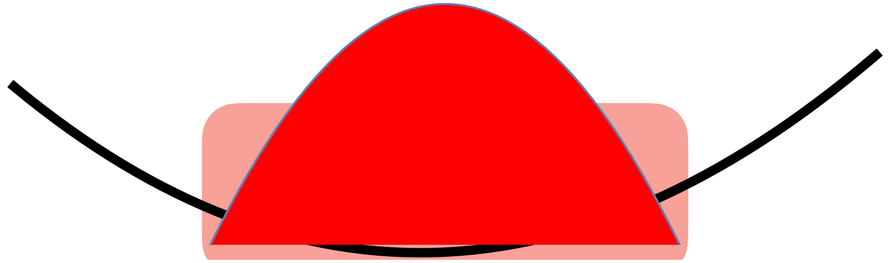
A mystery brought by 2D flat box potential

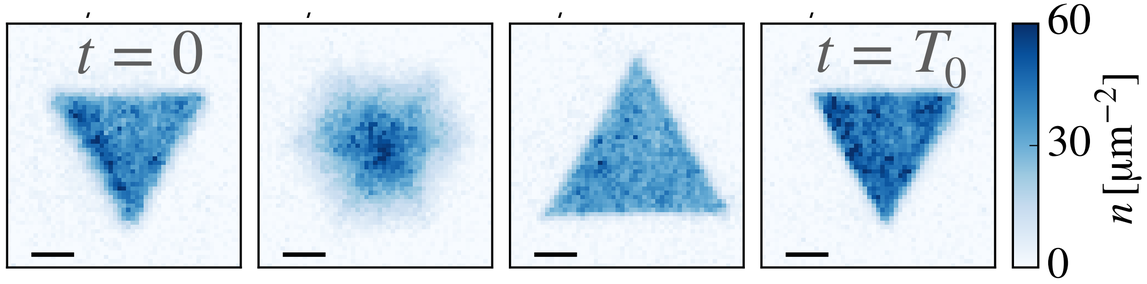
Saint-Jalm et al., Phys. Rev. X 9, 021035 (2019)



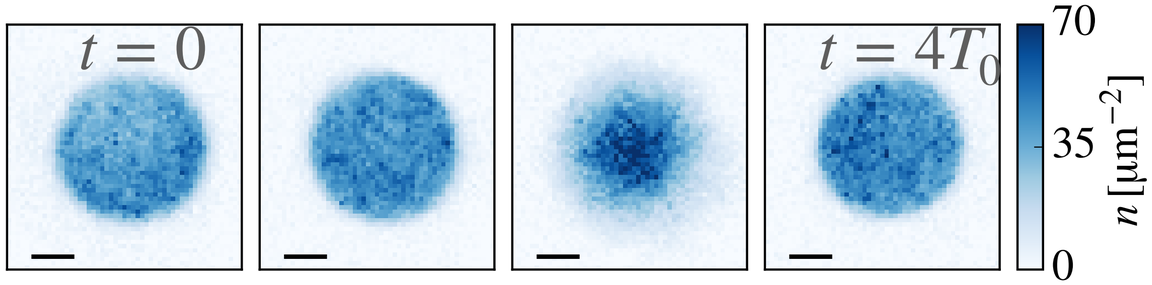
Uniform density, arbitrary shape







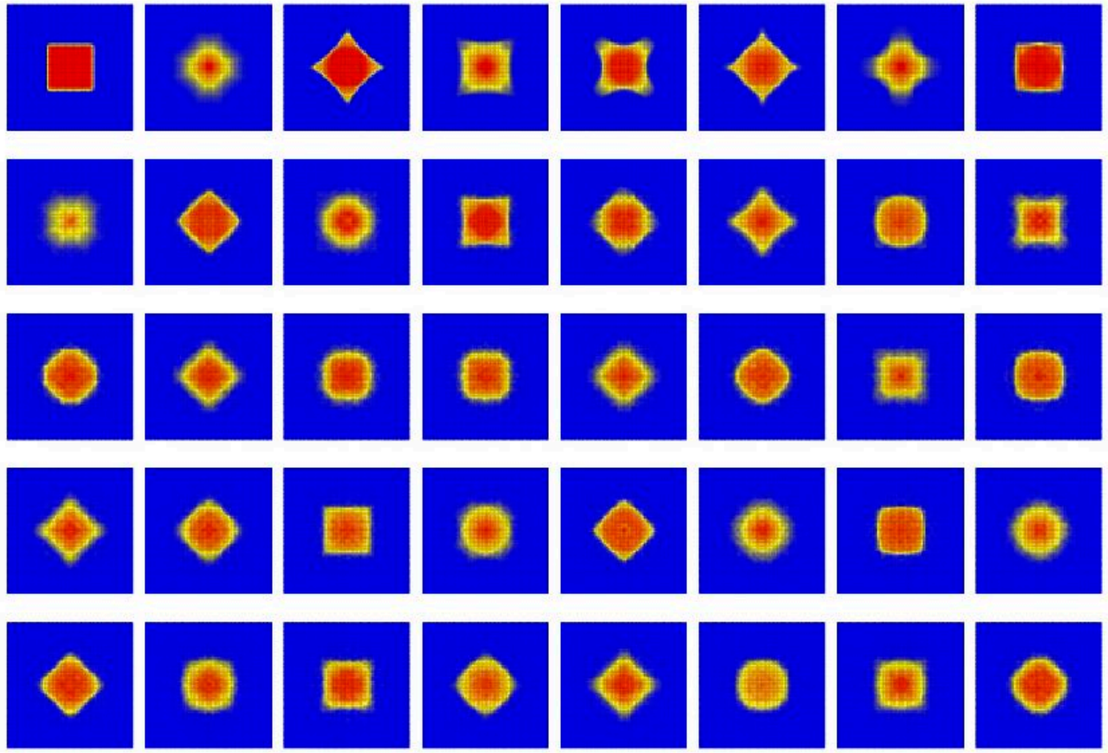
$T = T_0$ Agree with the well-established results

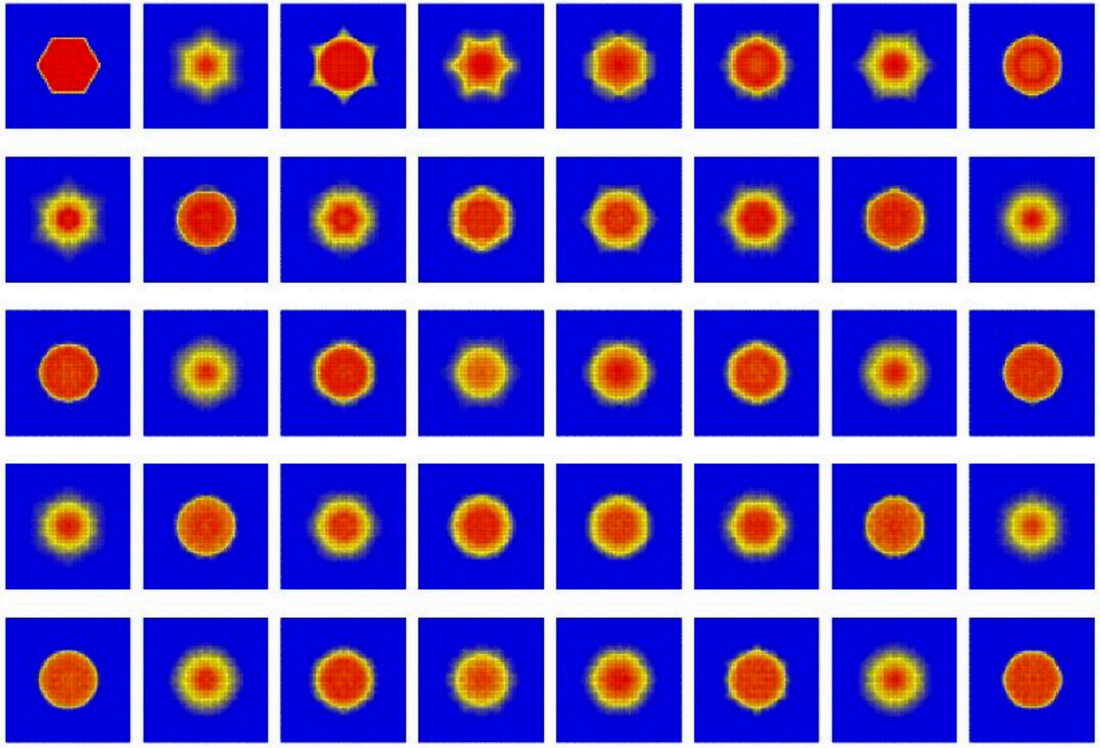


$T = 4T_0$?

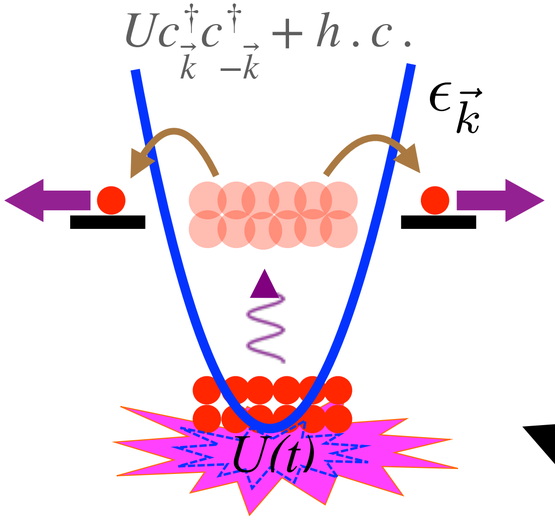
Other initial shapes: absence of periodicity

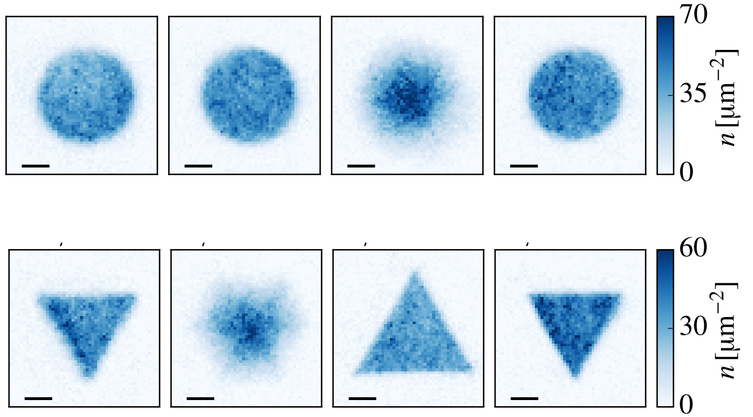
???

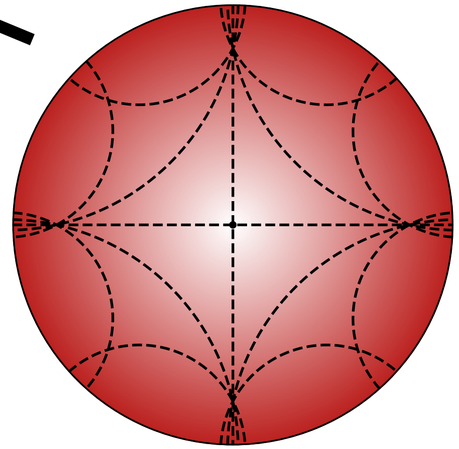




Distinct phenomena

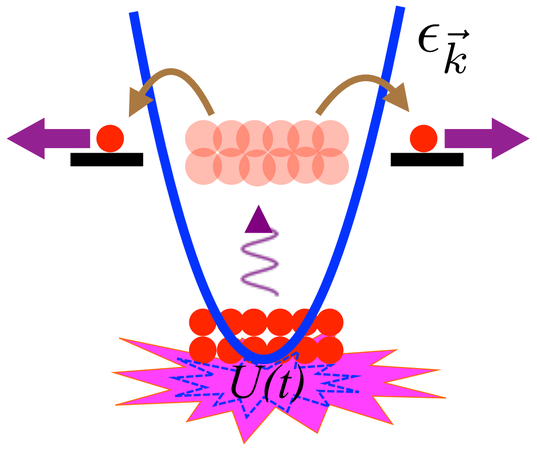






$$ds^2 = \frac{4(dx^2 + dy^2)}{1 - x^2 - y^2}$$

Poincaré disk underlying both



$$H = \sum_{\vec{k}} \epsilon_k c_k^\dagger c_k + U\cos(\omega t) \sum_{\vec{k}} c_{\vec{k}}^\dagger c^\dagger_{\ -}$$

$$K_0 = \frac{1}{2}(c_{\vec{k}}^\dagger c_{\vec{k}} + c_{-\vec{k}} c^\dagger_{\ -\vec{k}})$$

$$K_1 = \frac{1}{2}(c_{\vec{k}}^\dagger c^\dagger_{\ -\vec{k}} + c_{\vec{k}} c_{-\vec{k}})$$

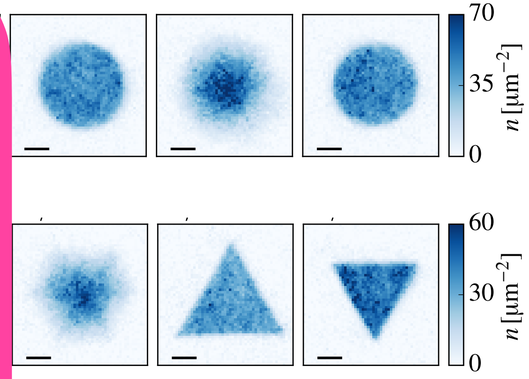
$$K_2 = \frac{1}{2i}(c_{\vec{k}}^\dagger c^\dagger_{\ -\vec{k}} - c_{\vec{k}} c_{-\vec{k}})$$

$$[K_1, K_2] = -iK_0$$

$$[K_0, K_1] = iK_2$$

$$[K_2, K_0] = iK_1$$

$$H = \xi_0 K_0 + \xi_1 K_1 + \xi_2 K_2$$



Length scale

$$l = \sqrt{\hbar/(M\omega)}$$

Energy scale

$$\hbar\omega$$

$$= \frac{1}{2}\left[\sum_i (-\frac{1}{2}\nabla_i^2 + \frac{1}{2}r_i^2) + \sum_{i<j} g\delta(\vec{r}_i - \vec{r}_j)\right]$$

$$= \frac{1}{2}\left[\sum_i (-\frac{1}{2}\nabla_i^2 + \frac{1}{2}r_i^2) + \sum_{i<j} g\delta(\vec{r}_i - \vec{r}_j)\right]$$

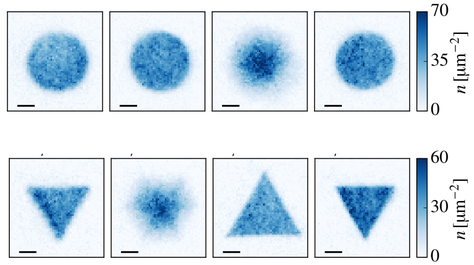
$$= \frac{1}{2}\left[\sum_i (-\frac{1}{2}\nabla_i^2 - \frac{1}{2}r_i^2) + \sum_{i<j} g\delta(\vec{r}_i - \vec{r}_j)\right]$$

$$K_2 = \frac{1}{4i}\left(\sum_i \vec{r}_i \cdot \nabla_i + \nabla_i \cdot \vec{r}_i\right)$$

The same SU(1,1) group, different representations

Poincaré disk:

Solving the mystery about breathers



C Lv, R Zhang, Q Zhou
PRL 125, 253002 (2020)

SU(1,1) Casimir operator

$$C = K_0^2 - K_1^2 - K_2^2$$

$$C\,|k,n\rangle = k(k-1)\,|k,n\rangle$$

$$K_0\,|k,n\rangle = (k+n)\,|k,n\rangle$$

SU(2) angular momentum

$$S^2 = S_x^2 + S_y^2 + S_z^2$$

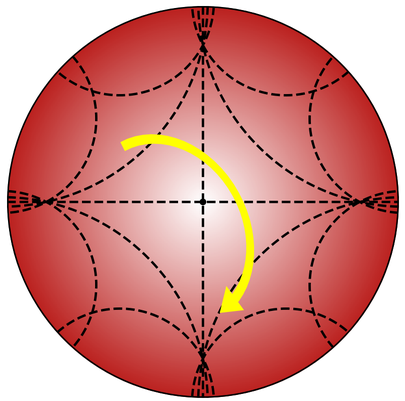
$$S^2\,|S,m\rangle = S(S+1)\,|S,m\rangle$$

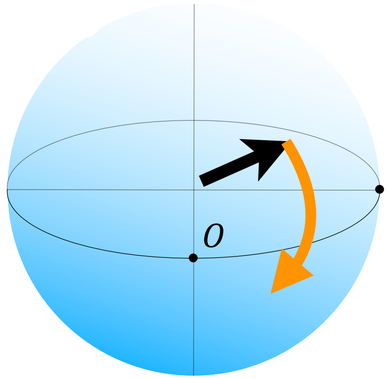
$$S_z\,|S,m\rangle = m\,|S,m\rangle$$

$K_{0,1,2}$ do not change k

A single Poincaré disk: a unique k

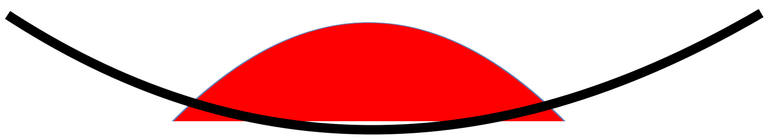
$S_{x,y,z}$ do not change S



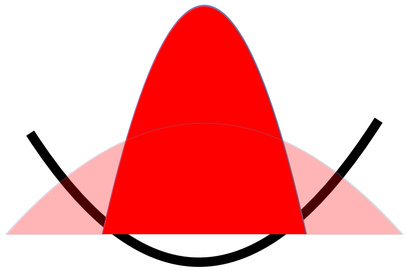


Traditional setup

$$V'(x) = \frac{1}{2} M\omega'^2 x^2$$



$$V(x) = \frac{1}{2} M\omega^2 x^2 = \kappa V'(x)$$



Initial state : an eigenstate of

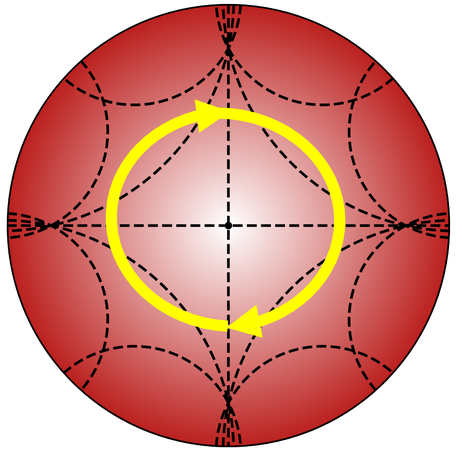
$$K_0' = KI + IN + PE' = (\frac{1}{2} + \frac{1}{2\kappa})K_0 + (\frac{1}{2} - \frac{1}{2\kappa})K_1$$

$$K_0 = KI + IN + \kappa PE'$$
$$K_1 = KI + IN - \kappa PE'$$

Also an eigenstate of C

$$|\Psi(0)\rangle = \sum_n c_n |k, n\rangle$$

A single k

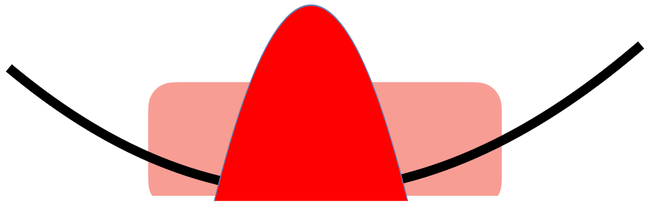


a trajectory on a **single** Poincare disk

$$T = T_0 = \pi/\omega$$

Flat box potential

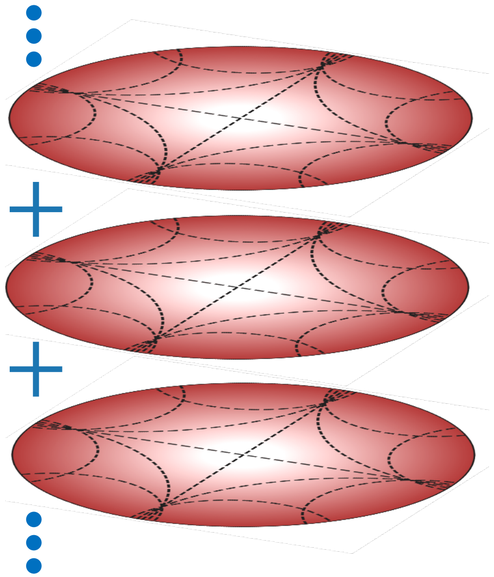


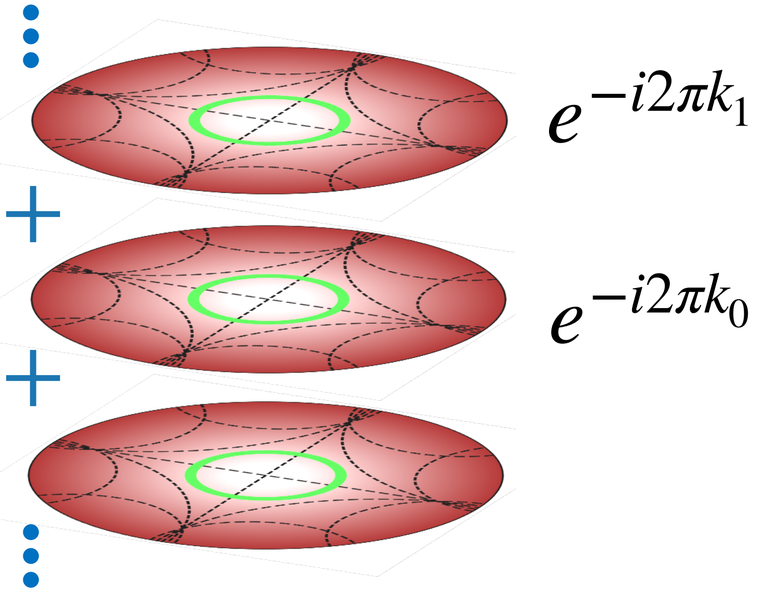


Initial state : no longer an eigenstate of $K_{i=0,1,2}$

$$|\Psi(0)\rangle = \sum_{k,n} c_{k,n} |k,n\rangle$$

$$|\Psi(t)\rangle = e^{-iK_0 t} |\Psi(0)\rangle = \sum_{k,n} c_{k,n} e^{-iK_0 t} |k,n\rangle$$





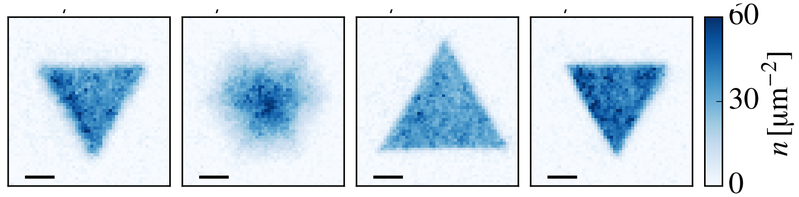
$e^{-i2\pi k_1}$

$e^{-i2\pi k_0}$

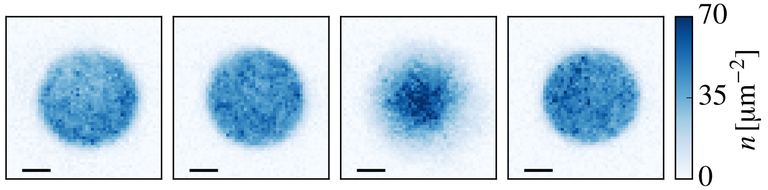
Different shape:different superposition



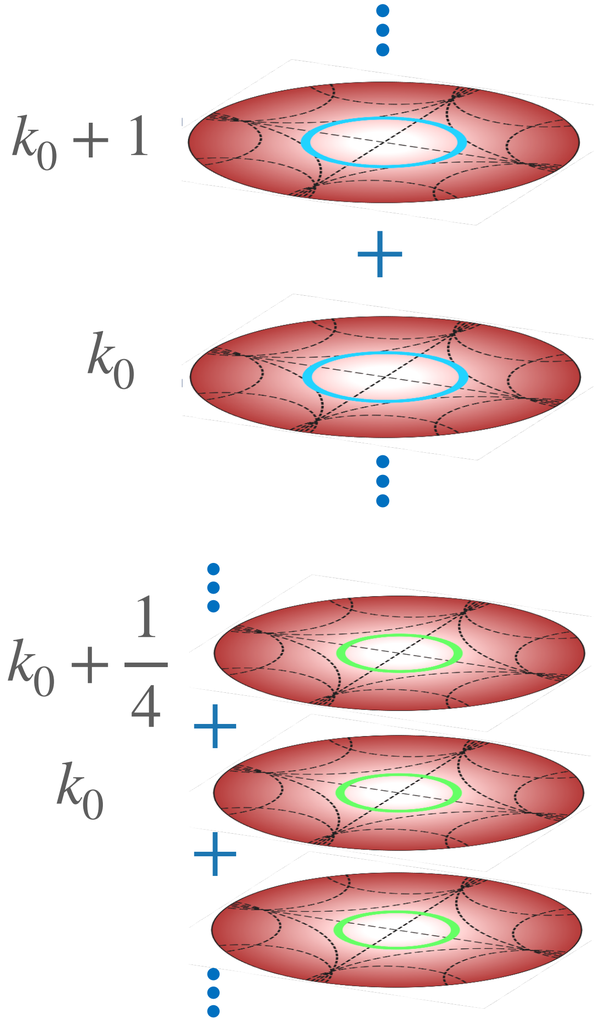
$$|\Psi(0)\rangle = \sum_{k,n} c_{k,n} |k, n\rangle$$

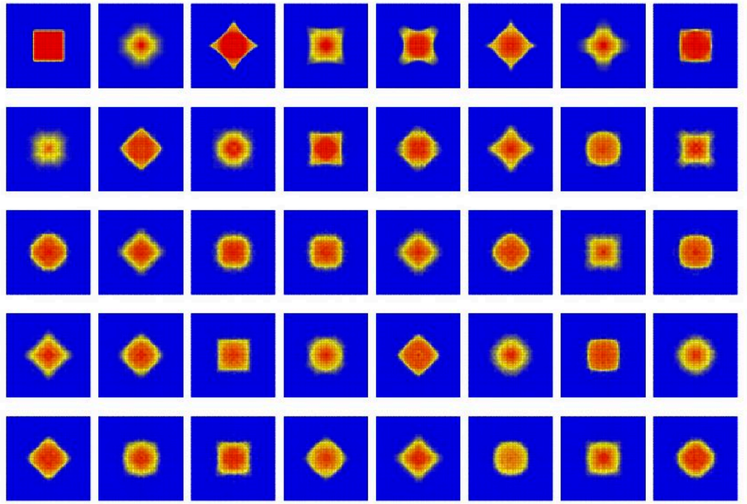


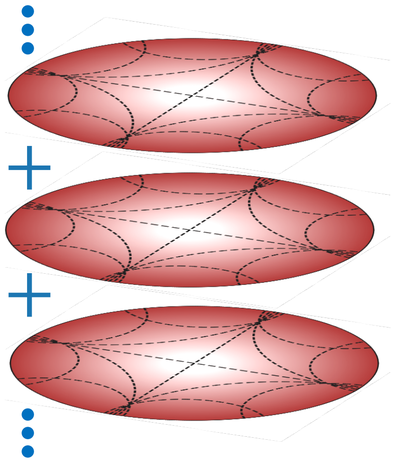
$T = T_0$



$T = 4T_0$





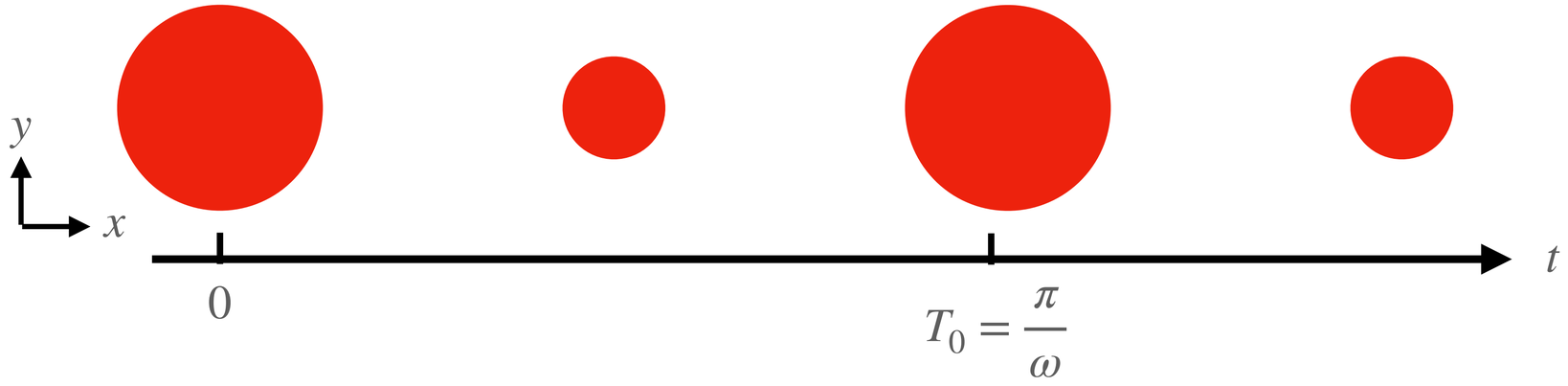


Incommensurate

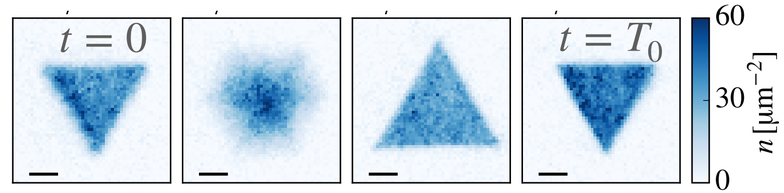
absence of periodicity

There is no mystery

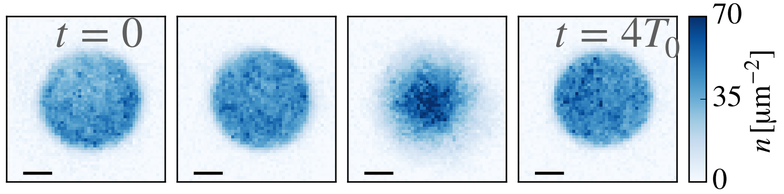
Y. Kagan, et al., Phys. Rev. A 54, 1753 (1996) ; Pitaevskii & Rosch, Phys. Rev. A 55, R853 (1997)…



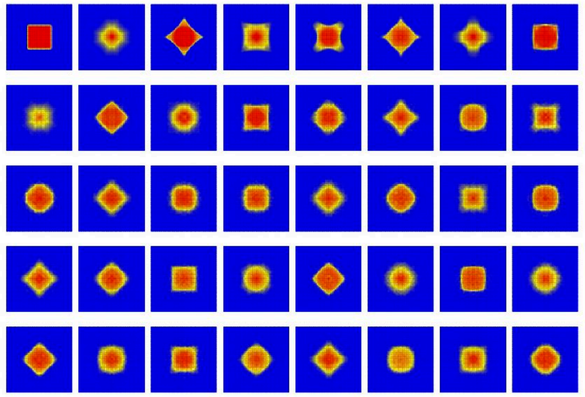
A special case:
a special class of
initial states



$T = T_0$

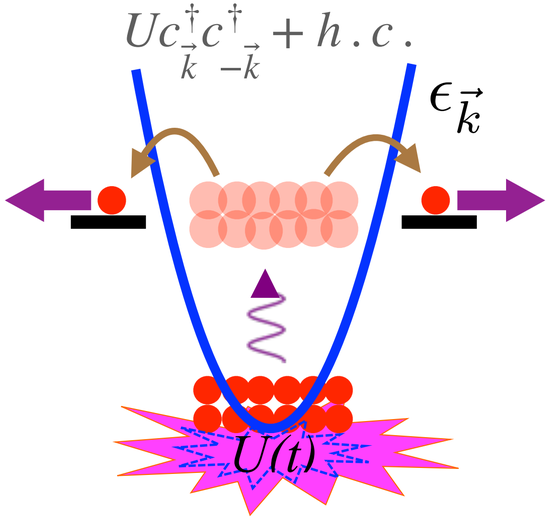


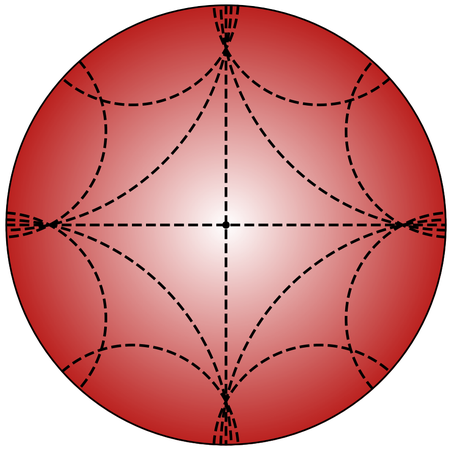
$T = 4T_0$

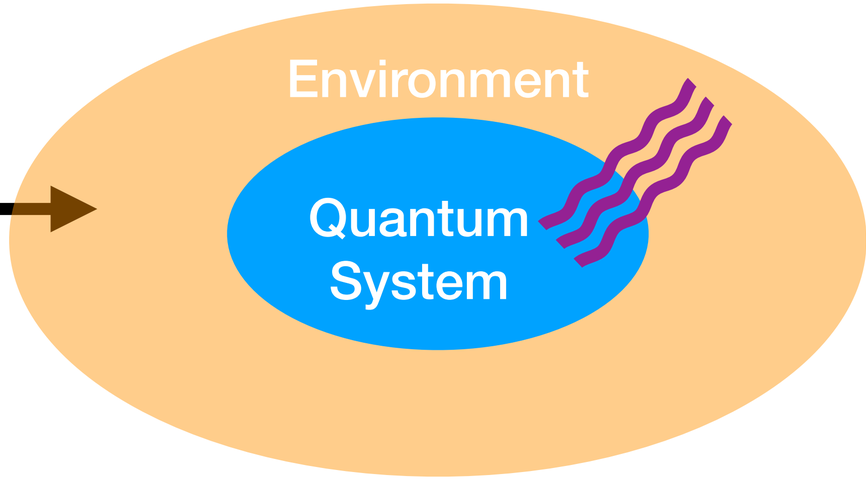


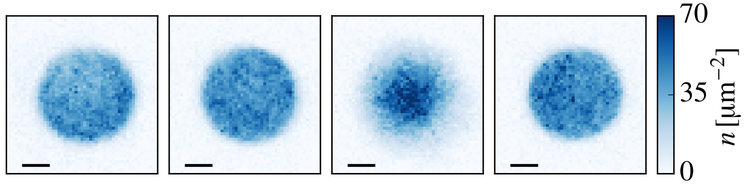
More generic!

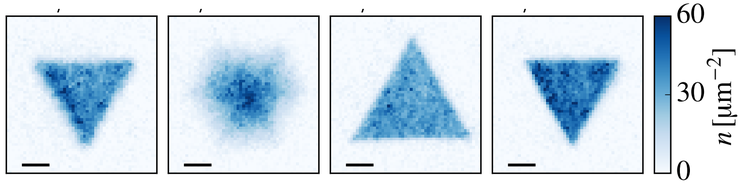
Geometry : Unifying distinct quantum dynamics











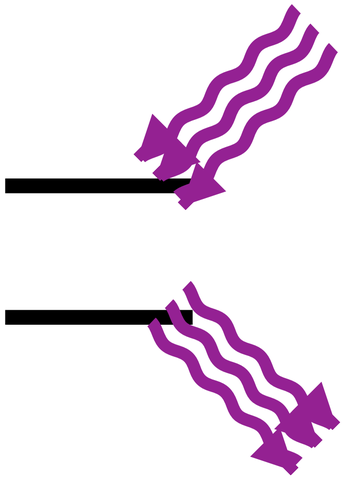
Unitary fermion gases

Periodically driven lattices

Spinor condensates

Efimov states …

non-Hermitian quantum dynamics



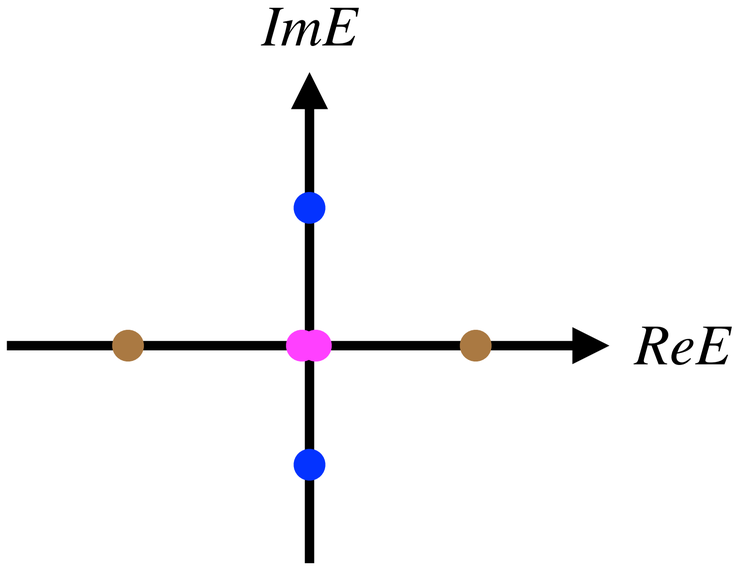
$$H = i\gamma S_z = \begin{pmatrix} i\gamma & \\ & -i\gamma \end{pmatrix}$$

More generically $H = iB_z S_z + iB_x S_x + B_y S_y$

PT symmetry unbroken $B_y^2 > B_x^2 + B_z^2$

Exceptional point $B_y^2 = B_x^2 + B_z^2$

PT symmetry breaking $B_y^2 < B_x^2 + B_z^2$



Bender, Boettcher, PRL. 80, 5243 (1998); Ashida, Gong, UedaAdvances in Physics 69, 3 (2020)…

Poincaré disk: Unifying Hermitian and non-hermitian quantum dynamics

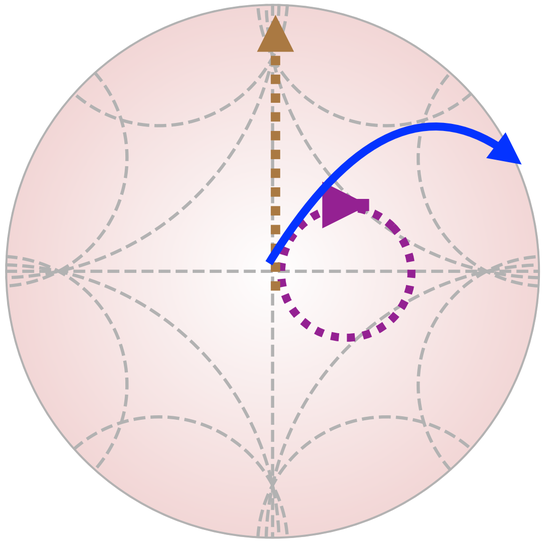
$$H = iB_z S_z + iB_x S_x + B_y S_y$$

$$[S_x, S_y] = iS_z$$
$$[S_y, S_z] = iS_x$$
$$[S_z, S_x] = iS_y$$

$$S_z = iK_1$$
$$S_x = iK_2$$
$$S_y = K_0$$

$$H = \xi_0 K_0 + \xi_1 K_1 + \xi_2 K_2$$

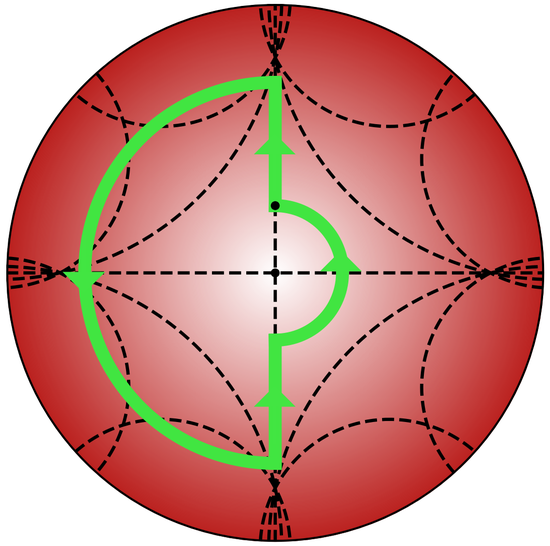
$$[K_1, K_2] = -iK_0$$
$$[K_0, K_1] = iK_2$$
$$[K_2, K_0] = iK_1$$



PT symmetry breaking

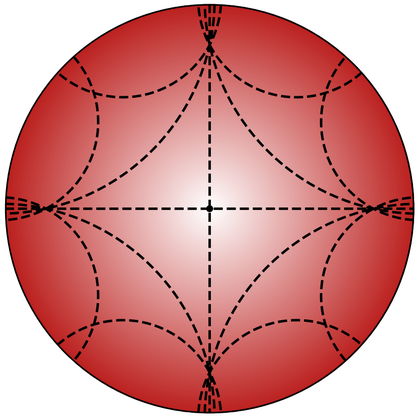
Exceptional point
=onsite of dynamical instability

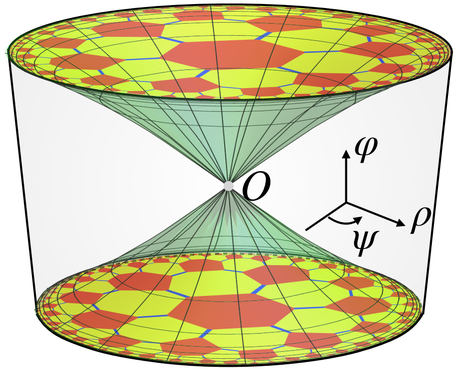
PT symmetry unbroken



Echoes in non-Hermitian systems

Including the overall phase of the wavefunction



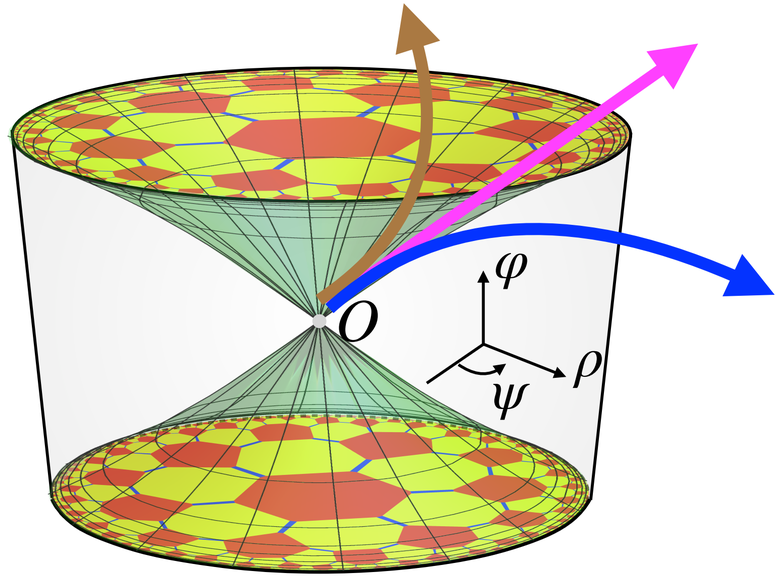


$\eta = 2\text{arctanh}(\rho)$

AdS_{2+1}

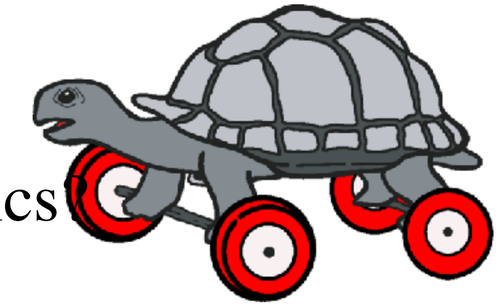
$$ds^2 = \mathrm{d}\eta^2 - \cosh^2(\eta)\mathrm{d}\varphi^2 + \sinh^2(\eta)\mathrm{d}\psi^2$$

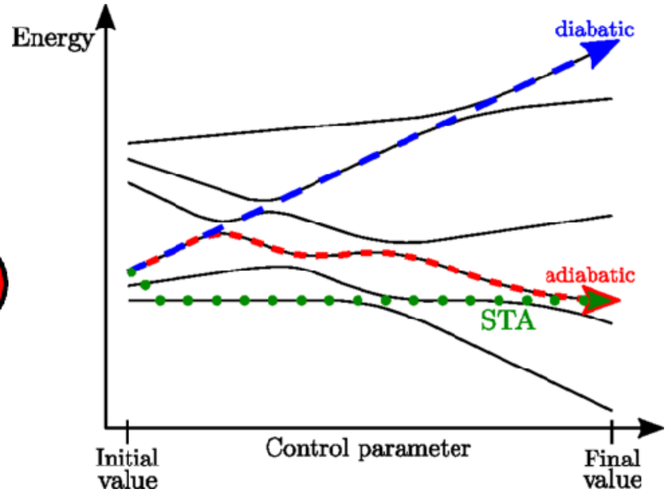
	$\xi^2 < 0$	$\xi^2 = 0$	$\xi^2 > 0$
AdS_{2+1}	**Spacelike**	**Light cone**	**Timelike**
Spin in complex B	PT symmetry breaking	Exceptional point	PT symmetry unbroken
Driven BEC/ 2D breather/ unitary fermions	Dynamically unstable	Transition point	Dynamically stable



Application: shortcuts to adiabaticity

A key question： how to access fastest dynamics





Answer from geometry: shortest path

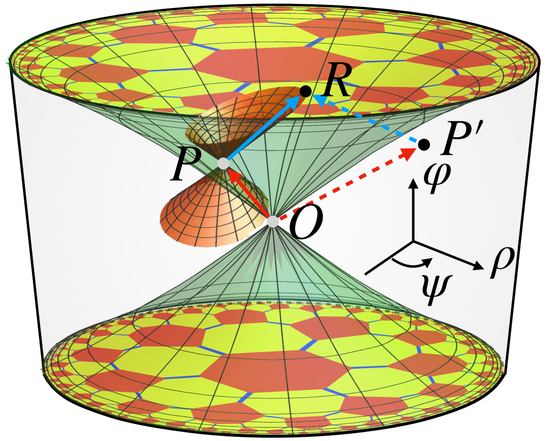
In AdS, moving on light cones S=0

More generically

$$H_1(\delta) = \hbar\omega_0\left[(\delta^{-1}-\delta)K_0 + (\delta^{-1}+\delta)K_1\right] \qquad 0 < \tau < \tau_1$$

$$H_2 = \frac{i\hbar}{t_2}\log\left(U_{\rm tar}e^{iH_1(\delta)\tau_1/\hbar}\right) \qquad \tau_1 < \tau < \tau_1+\tau_2$$

$$\delta \to 0 \qquad \tau_1 = T_1\delta \qquad S -> 0$$



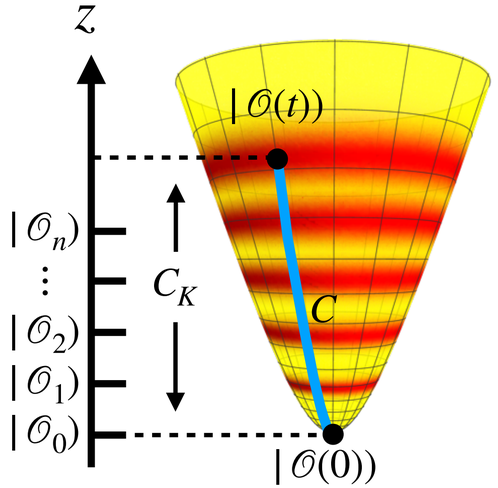
Building Krylov complexity from circuit complexity

$$i\partial_t \hat{\mathcal{O}} = i[\hat{H}, \hat{\mathcal{O}}]$$

$$|\mathcal{O}(t)) = \sum_n \phi_n(t)|\mathcal{O}_n)$$

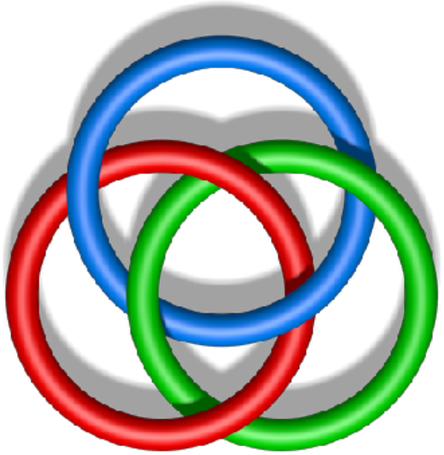
$$C_K(t) = \sum_n n|\phi_n(t)|^2$$

Parker, et al., PRX 9, 041017 (2019); Caputa, et al., PRR 4, 013041 (2022)….



Lv. Zhou, arXiv:2303.07343

Symmetry



Dynamics **Geometry**

- Visualizing and understanding quantum dynamics
- New quantum controls and quantum sensing
- Unifying Hermitian and non-Hermitian phenomena
- Bridge multiple subfields

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